

# A GENERALIZATION OF POWERS-STØRMER INEQUALITY

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ABSTRACT. Let  $A, B$  be the positive semidefinite matrices. A matrix version of the famous Powers-Størmer's inequality

$$2\text{Tr}(A^\alpha B^{1-\alpha}) \geq \text{Tr}(A + B - |A - B|), \quad 0 \leq \alpha \leq 1,$$

was proven by Audenaert et. al. We establish a comparison of eigenvalues for the matrices  $A^\alpha B^{1-\alpha}$  and  $A + B - |A - B|$ ,  $0 \leq \alpha \leq 1$ , subsuming the Powers-Størmer's inequality. We also prove several related norm inequalities.

## 1. INTRODUCTION

Let  $M_n$  denote the algebra of all  $n \times n$  complex matrices. A Hermitian member  $A$  of  $M_n$  with all non-negative eigenvalues is known as positive semi-definite matrix, simply denoted by  $A \geq 0$ . We shall denote by  $P_n$ , the collection of all such matrices. For  $A, B$  Hermitian in  $M_n$ , we employ the positive semi-definite ordering:  $A \geq B$  if and only if  $A - B \geq 0$ . By  $|A|$ , we mean the positive square root of the matrix  $A^*A$ , i.e.,  $(A^*A)^{1/2}$ . The Jordan decomposition of a Hermitian matrix  $A$  is given by  $A = A_+ - A_-$ , where  $A_+$  and  $A_-$  are the members of the  $P_n$  along with  $A_+A_- = 0$  (see [3], page 99). We shall consider  $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \lambda_n(A) \geq 0$ , the eigenvalues of  $A \in P_n$ , arranged in decreasing order and repeated according to their multiplicity. Similarly  $s_1(A) \geq s_2(A) \geq \cdots s_n(A) \geq 0$ , denote the singular values (eigenvalues of  $|A|$ ) of a matrix  $A \in P_n$ , arranged in decreasing order and repeated according to their multiplicity. By  $|||\cdot|||$ , we mean any unitarily invariant norm, while  $||\cdot||$  denotes operator norm on  $M_n$ .

In 2007, Audenaert et. al. [1] solved a long standing open problem to identify the classical quantum Chernoff bound in the area of information theory. After the mathematical formulation of that problem, they proved a nontrivial and fundamental inequality relating to the trace distance to the quantum Chernoff bound. That became a key result to a solution of the problem and is stated as follows:

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Let  $A, B$  be positive matrices and  $0 \leq \alpha \leq 1$ . Then

$$2\text{Tr}(A^\alpha B^{1-\alpha}) \geq \text{Tr}(A + B - |A - B|) \quad (1.1)$$

holds. A particular case  $\alpha = 1/2$  in (1.1) is a well known Powers-Størmer's inequality [7], which was proved in 1970. For such literature and detail of inequalities the reader may refer [6]. Subsequently in 2008, again Audenaert et. al. [2] worked on symmetric as well as with asymmetric quantum hypothesis testing. In [2] also, they proved some similar type of inequalities as that of (1.1) which played a key role in getting the optimal solution to the symmetric classical hypothesis test.

In 2011, Y. Ogata [5] generalised the Powers-Størmer inequality to von Neumann algebras. Recently several authors including D. Hoa et. al. [4] generalised this inequality on  $C^*$ -algebras using the technique of operator monotone functions on  $[0, \infty)$ .

We aim to prove the comparison of eigenvalues of  $A+B-|A-B|$  and  $2A^\alpha B^{1-\alpha}$ , generalizing all the forms of Powers-Størmer's inequality. We shall also prove several other associated norm inequalities.

## 2. MAIN RESULTS

**Lemma 2.1.** *Let  $A, B \in P_n$  then there exist a matrix  $S \in P_n$  satisfying*

- (1)  $S \leq A, S \leq B$
- (2) *if  $T \leq A, T \leq B$ , is a fixed Hermitian matrix then  $\lambda_i(T) \leq \lambda_i(S)$  for  $1 \leq i \leq n$ .*

*Proof.* We shall first prove this result for either of  $A$  or  $B$  invertible. So assume  $B$  is invertible i.e.  $B$  is Hermitian and whose all the eigenvalues are positive. As is well-known that  $B^{-1/2}AB^{-1/2} \in P_n$  and so unitarily diagonalizable. We assume that  $B^{-1/2}AB^{-1/2} = U^*DU$  for some  $U$  a unitary and  $D$  a diagonal matrix with diagonal entries as  $d_1 \geq d_2 \geq d_3 \cdots \geq d_n \geq 0$ . Choose  $S = B^{1/2}U^*D_1UB^{1/2}$ , where  $D_1$  is a diagonal matrix with diagonal entries as  $t_1 \geq t_2 \geq t_3 \cdots \geq t_n \geq 0$ , such that  $t_i = \min\{d_i, 1\}$ . This choice of  $S$  satisfies

$$S = B^{1/2}U^*D_1UB^{1/2} \leq B^{1/2}U^*DUB^{1/2} = A,$$

$$S = B^{1/2}U^*D_1UB^{1/2} \leq B^{1/2}U^*IUB^{1/2} = B.$$

For (2), let  $T \leq A$  as well as  $T \leq B$  be a fixed Hermitian matrix, then by Weyl's monotonicity principle we have  $\lambda_i(T) \leq \lambda_i(A)$  and  $\lambda_i(T) \leq \lambda_i(B)$  for all  $i = 1, 2, \dots, n$ . If

$$\lambda_i(T) \leq \lambda_i(S) \quad \text{for } 1 \leq i \leq n,$$

the above construction of  $S$  meets both the requirements.

If

$$\lambda_j(T) \geq \lambda_j(S) \quad \text{for some } 1 \leq j \leq n,$$

then we replace that particular  $t_j$  with  $\lambda_j(T)$  in  $D_1$ . Then, this choice of  $S$  meets both the requirements.

The general case follows by using continuity argument.  $\square$

Now onwards, we shall denote  $S$  by  $\min\{A, B\}$ .

**Theorem 2.2.** *Let  $A, B \in P_n$  then*

$$\lambda_i(A + B - |A - B|) \leq 2\lambda_i(A^\alpha B^{1-\alpha}) \quad (2.1)$$

for  $0 \leq \alpha \leq 1$  and  $1 \leq i \leq n$ .

*Proof.* Let  $T$  be any Hermitian matrix with Jordan decomposition  $T_+ - T_-$ . Then,  $|T| = T_+ + T_-$ , so  $T - |T| = -2T_- \leq 0$ . Using this fact for  $A - B$ , we can write,

$$A + B - |A - B| = 2(B - (A - B)_-) \leq 2B. \quad (2.2)$$

Replacing  $B$  by  $A$  in above inequality, we obtain

$$A + B - |A - B| = 2(A - (B - A)_-) \leq 2A. \quad (2.3)$$

Now, on using Lemma 2.1, we obtain

$$\begin{aligned} \lambda_i(A + B - |A - B|) &\leq 2\lambda_i(\min\{A, B\} = S) \\ &\leq 2\lambda_i(S^{\alpha/2} B^{1-\alpha} S^{\alpha/2}), \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

To complete the proof, it is enough to show

$$\lambda_i(S^{\alpha/2} B^{1-\alpha} S^{\alpha/2}) \leq \lambda_i(A^\alpha B^{1-\alpha}), \quad \text{for } 1 \leq i \leq n.$$

Indeed,

$$\begin{aligned} 2\lambda_i(S^{\alpha/2} B^{1-\alpha} S^{\alpha/2}) &= 2\lambda_i(B^{(1-\alpha)/2} S^\alpha B^{(1-\alpha)/2}) \\ &\leq 2\lambda_i(B^{(1-\alpha)/2} A^\alpha B^{(1-\alpha)/2}) \\ &= 2\lambda_i(A^\alpha B^{1-\alpha}) \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

$\square$

**Corollary 2.3.** (Cf. [1, 2], Theorem 1, Theorem 2) Let  $A, B \in P_n$  then for  $0 \leq \alpha \leq 1$

$$0 \leq \text{Tr}(A + B - |A - B|) \leq 2\text{Tr}(A^\alpha B^{1-\alpha}). \quad (2.4)$$

*Proof.* Let  $A - B = (A - B)_+ - (A - B)_-$  be the Jordan decomposition of  $A - B$ , then for  $1 \leq i \leq n$ ,

$$\lambda_i(A - B)_- \leq \lambda_i(B), \quad (2.5)$$

(see Lemma IX.4.1 of [3]). The first inequality from the left side in (2.4) follows immediately from (2.2) and (2.5). The last inequality follows from Theorem 2.2.  $\square$

**Corollary 2.4.** *Let  $A, B \in P_n$  then for  $0 \leq \alpha \leq 1$*

- (i)  $|||(A + B - |A - B|)_+||| \leq 2|||A^\alpha B^{1-\alpha}|||$
- (ii)  $|||(A + B - |A - B|)_-||| \leq 2|||A^\alpha B^{1-\alpha}|||.$

*Proof.* (i) As  $A, B \in P_n$ , hence, without loss of generality we assume

$$\begin{aligned} \lambda_1(A + B - |A - B|) &\geq \lambda_2(A + B - |A - B|) \geq \cdots \geq \lambda_k(A + B - |A - B|) \geq 0 \\ &> \lambda_{k+1}(A + B - |A - B|) \geq \cdots \geq \lambda_n(A + B - |A - B|) \end{aligned}$$

and

$$\lambda_1(A^{\alpha/2} B^{1-\alpha} A^{\alpha/2}) \geq \lambda_2(A^{\alpha/2} B^{1-\alpha} A^{\alpha/2}) \geq \cdots \geq \lambda_n(A^{\alpha/2} B^{1-\alpha} A^{\alpha/2}) \geq 0.$$

The matrix  $A + B - |A - B|$  is Hermitian, so unitarily diagonalizable, i.e.,

$$A + B - |A - B| = W^* D_2 W,$$

for  $W$  a unitary matrix and  $D_2$  a diagonal matrix given by

$$D_2 = \text{diag}(\lambda_1(A + B - |A - B|), \dots, \lambda_n(A + B - |A - B|)).$$

Now, using Jordan decomposition of  $A + B - |A - B|$ , (see [3], page 99) provides that

$$(A + B - |A - B|)_+ = W^* D_{2+} W \quad \text{and} \quad (A + B - |A - B|)_- = W^* D_{2-} W,$$

where  $D_{2+}$  and  $D_{2-}$  are diagonal matrices in  $P_n$ , given by

$$D_{2+} = \text{diag}(\lambda_1(A + B - |A - B|), \dots, \lambda_k(A + B - |A - B|), 0, \dots, 0)$$

and

$$D_{2-} = \text{diag}(0, \dots, -\lambda_{k+1}(A + B - |A - B|), \dots, -\lambda_n(A + B - |A - B|)).$$

By the above discussion, we clearly obtain

$$\lambda_i(A + B - |A - B|)_+ = \begin{cases} \lambda_i(A + B - |A - B|), & \text{for } i = 1, 2, \dots, k \\ 0, & \text{for } i = k + 1, k + 2, \dots, n, \end{cases}$$

and

$$\lambda_i(A + B - |A - B|)_- = \begin{cases} 0, & \text{for } i = 1, 2, \dots, k \\ -\lambda_i(A + B - |A - B|), & \text{for } i = k + 1, k + 2, \dots, n. \end{cases}$$

Now, using inequality (2.1) alongwith  $\lambda_i(A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}) = \lambda_i(A^\alpha B^{1-\alpha})$ , we obtain

$$\lambda_i((A + B - |A - B|)_+) \leq 2\lambda_i(A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}), \quad \text{for } i = 1, 2, \dots, n,$$

i.e,

$$s_i((A + B - |A - B|)_+) \leq 2s_i(A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}), \quad \text{for } i = 1, 2, \dots, n. \quad (2.6)$$

On using Theorem IV.2.2 and then Proposition IX.1.1 of [3] in (2.6), we obtain

$$\begin{aligned} |||(A + B - |A - B|)_+||| &\leq 2|||A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}||| \\ &\leq 2|||A^\alpha B^{1-\alpha}|||. \end{aligned} \quad (2.7)$$

This completes the proof of (i).

For a proof of (ii), use (2.2) and (2.5) to obtain,

$$\lambda_i((A + B - |A - B|)_-) \leq 2\lambda_i(B). \quad (2.8)$$

Now, replace  $B$  by  $A$  in (2.8), we obtain,

$$\lambda_i((A + B - |A - B|)_-) \leq 2\lambda_i(A). \quad (2.9)$$

Again, on using similar technique as in Theorem 2.2, we get the desired result.  $\square$

The following corollary is an immediate consequence of Corollary 2.4.

**Corollary 2.5.** *Let  $A, B \in P_n$  then for  $0 \leq \alpha \leq 1$*

$$||A + B - |A - B||| \leq 2||A^\alpha B^{1-\alpha}||. \quad (2.10)$$

*Proof.* The operator norm for any Hermitian matrix  $T$  is given by

$$||T|| = \max\{||T_+||, ||T_-||\}.$$

Using the above fact for the matrix  $A + B - |A - B|$  and Corollary 2.4 to obtain (2.10).  $\square$

**Theorem 2.6.** *Let  $A, B \in P_n$  then for  $0 \leq \alpha \leq 1$ , some projection  $P$  and  $\beta \geq 0$ ,*

$$|||A + B - |A - B||| \leq 2|||A^\alpha B^{1-\alpha} - \beta A^{\alpha/2} P A^{-\alpha/2}|||. \quad (2.11)$$

*Proof.* Let  $X = \text{diag}(x_1, x_2, \dots, x_n)$  and  $T = \text{diag}(t_1, t_2, \dots, t_n)$  be the matrices comprised of  $x'_i$ s and  $t'_i$ s as eigenvalues of  $A + B - |A - B|$  and  $2A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}$  in decreasing order respectively. Using Theorem (2.2) on  $X$  and  $T$ , we get

$$x_i \leq t_i \quad \text{for } i = 1, 2, \dots, n.$$

If  $\beta = \text{Tr}(T) - \text{Tr}(X)$ , then on using Corollary 2.3, we obtain  $\beta \geq 0$ . Consider

$$T_1 = 2A^{\alpha/2}B^{1-\alpha}A^{\alpha/2} - \beta Q_n,$$

where  $\sum_{i=1}^n t_i Q_i$  is the spectral decomposition of  $2A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}$ . It is clear from the construction of  $T_1$  that eigenvalues of  $T_1$  are all same and in the same order as that of  $2A^{\alpha/2}B^{1-\alpha}A^{\alpha/2}$  except the last one. So we may assume  $(t_1, t_2, \dots, t_{n-1}, \gamma_n)^t$  as a column vector of eigenvalues of  $T_1$ , satisfying

$$\sum_{i=1}^k x_i \leq \sum_{i=1}^k t_i \quad \text{for } k = 1, 2, 3, \dots, n-1,$$

and

$$\sum_{i=1}^n x_i = \sum_{i=1}^{n-1} t_i + \gamma_n.$$

Finally, using Example II.3.5 in [3], we get

$$\sum_{i=1}^k |x_i| \leq \sum_{i=1}^k t_i \quad \text{for } k = 1, 2, \dots, n-1,$$

and

$$\sum_{i=1}^n |x_i| \leq \sum_{i=1}^{n-1} t_i + |\gamma_n|.$$

Equivalently,

$$\sum_{i=1}^k s_i(A + B - |A - B|) \leq \sum_{i=1}^k s_i(T_1) \quad \text{for } k = 1, 2, \dots, n.$$

Hence,

$$\begin{aligned} |||A + B - |A - B||| &\leq |||2A^{\alpha/2}B^{1-\alpha}A^{\alpha/2} - \beta Q_n||| \\ &= |||A^{-\alpha/2}(2A^{\alpha}B^{1-\alpha}A^{\alpha/2} - \beta A^{\alpha/2}Q_n)||| \\ &\leq |||2A^{\alpha}B^{1-\alpha} - \beta A^{\alpha/2}Q_n A^{-\alpha/2}|||, \end{aligned}$$

using Theorem IV.2.2 of [3] for the first inequality and Proposition IX.1.1 of [3] for the second inequality. This completes the proof.

□

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